ANOVA

Lectures 22-23

Readings: GW 13

Review Independent Samples *t* test

$H_0: \mu_P = \mu_T$ $\quad H_A: \mu_P \neq \mu_T$

$H_0: \mu_P - \mu_T = 0$ $\quad H_A: \mu_P - \mu_T \neq 0$

	Placebo	Treatment
	4	7
	5	6
	2	4
	6	5
	4	7
	3	5
	5	7
	6	
	4	
Mean	4.33	5.86
SS	14	8.86

Review Independent Samples *t* test

	Placebo	Treatment
	4	7
	5	6
	2	4
	6	5
	4	7
	3	5
	5	7
	6	
	4	
Mean	4.33	5.86
SS	14	8.86

$$t_{crit}(14) = 2.14$$

$$s_p^2 = \frac{SS_1 + SS_2}{df_1 + df_2}$$

$$t_{df_1+df_2} = \frac{\bar{X}_p - \bar{X}_t}{s_{\bar{X}_p - \bar{X}_t}} = \frac{\bar{X}_p - \bar{X}_t}{\sqrt{\left(\frac{s_p^2}{n_p} + \frac{s_p^2}{n_t}\right)}} =$$

$$C.I.: \left(\bar{X}_p - \bar{X}_t\right) \pm t_{crit}\left(s_{\bar{X}_p - \bar{X}_t}\right) =$$

Frequently Need to Compare More than Two Groups

- Success versus Failure Feedback on subsequent performance; If means differ is it because success feedback helps? Because failure feedback hurts? Both?
  - Need a no feedback control condition
- Is there an effect of a therapeutic intervention and does it depend on dosage?
  - Need for placebo and various dosage levels of treatment
- Does the effect of a therapeutic intervention depend upon prior symptom level?
  - Need for placebo and treatment conditions for participants who have high versus low levels of prior symptoms
- Why NOT successive pairwise *t's?*

Type I Error and Multiple Tests

With an α = .05, there is a 5% risk of a Type I error.

Thus, for every 20 hypothesis tests, you expect to make one Type I error.

The more tests you do, the more risk there is of a Type I error.

***Test-wise* alpha level**

The alpha level you select for each individual hypothesis test.

***Experiment-wise* alpha level**

The total probability of a Type I error accumulated from all of the separate tests in the experiment.

e.g. three t-tests, each at α = .05 -> an experimentwise alpha of .14

Two reasons for using ANOVA

1) Compare multiple means

2) Do so in a way that doesn't inflate the type-I error rate. ANOVA uses one test with one alpha level to evaluate all the mean differences, thereby avoiding the problem of an inflated experiment-wise type-I error

Research Designs

Like the t-test, ANOVA is used when the independent variable in a study is categorical (e.g. treatment groups) and the dependent variable is continuous.

A t-test is used when there are only two groups to compare. ANOVA is used when there are two or more groups

Research Designs

Terminology

In analysis of variance, an independent variable is called a *factor*. (e.g. treatment groups)

The groups that make up the independent variable are called *levels* of that factor. (e.g. low dose, high dose, control)

A research study that involves only one factor is called a *single-factor* design.

A research study that involves more than one factor is called a *factorial* design (e.g. treatment group and gender)

ANOVA Notation

H_o: $\mu_1 = \mu_2 = \mu_3 = \ldots\mu_k$

H_A: at least one population mean is different

- j = a particular group; in total there are k groups
- i = a particular person in the jth group, i varies between 1 and n_j (initially assume equal n_j)
- $k(n_j)=N$

X_{ij} $\quad \bar{X}_j$ $\quad \bar{\bar{X}}$ $\quad n_j$

ANOVA theory

(formulas for elucidation not memorization)

- If H_0 true, σ known, and n constant, then:

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} \qquad \sigma^2_{\bar{X}} = \frac{\sigma^2}{n} \qquad \frac{n\sigma^2_{\bar{X}}}{\sigma^2} = 1$$

- We will estimate this ratio (called the F ratio) and examine whether it is close to one. If it is, then we will not reject the null hypothesis; if it is sufficiently different from one then we will reject the null hypothesis.

$$F = \frac{n\sigma^2_{\bar{X}}}{\sigma^2} = \frac{MS_{between}}{MS_{within}}$$

ANOVA Intuition

Random variation + effect

t statistic

$$t = \frac{\text{obtained difference between sample means}}{\text{difference expected by chance}}$$

Random variation

F statistic

$$F = \frac{\text{obtained variance between sample means}}{\text{variance expected by chance}}$$

$$F = \frac{MS_{between}}{MS_{within}}$$

ANOVA Example

Suppose that a psychologist wants to examine learning performance under three temperature conditions: 50º, 70º, 90º. Three samples of subjects are selected, one sample for each treatment condition. The subjects are taught some basic material and given a short test.

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X} = 1$	$\bar{X} = 4$	$\bar{X} = 1$

Factor: temperature

Levels: 50, 70, 90

ANOVA Example

Why don't we just run three t-tests?

compare 50º sample and 70º sample
compare 50º sample and 90º sample
compare 70º sample and 90º sample

ANOVA Example

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$

There is some variability among the treatment means

Is it more than we might expect from chance alone?

The Logic of ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$

Quick review of how we calculate variance:

$$s_1^2 = \frac{SS_1}{n_1 - 1} \quad = \frac{SS_1}{df_1}$$

The Logic of ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$
$SS_1 = 6$	$SS_2 = 6$	$SS_3 = 4$

So what is our estimate of the random variability due to chance?

$$s_p^2 = MS_{within} = \frac{SS_1 + SS_2 + SS_3}{df_1 + df_2 + df_3}$$

This comes from the "within-treatment" variabilities

The Logic of ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$
$SS_1 = 6$	$SS_2 = 6$	$SS_3 = 4$

So what is our estimate of the random variability due to chance?

$$MS_{within} = \frac{SS_1 + SS_2 + SS_3}{df_1 + df_2 + df_3} = \frac{\sum_{j=1}^{k} SS_j}{N - k} = \frac{SS_{within}}{df_{within}}$$

The Logic of ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$
$SS_1 = 6$	$SS_2 = 6$	$SS_3 = 4$

So what is our estimate of the random variability due to chance?

$$MS_{within} = \frac{SS_{within}}{df_{within}}$$

A summary of how much each datapoint varies from its treatment mean. Exactly like pooled s^2 from 2-sample t-test, except 3+ groups.

The Logic of ANOVA

Treatment 1 50°	Treatment 2 70°	Treatment 3 90°
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$

F statistic

$$F = \frac{\text{obtained variance between sample means}}{\text{variance expected by chance}} = \frac{MS_{between}}{MS_{within}}$$

The Logic of ANOVA

Treatment 1 50°	Treatment 2 70°	Treatment 3 90°
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$

So what is the variance between our treatment means?

$$s_b^2 = MS_{between} = \frac{SS_{between}}{df_{between}}$$

This is the "between-treatment" variability. Bigger effect = greater variability.

The Logic of ANOVA

Treatment 1 50°	Treatment 2 70°	Treatment 3 90°
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$
$(\bar{X}_1 - \bar{\bar{X}}) = -1$	$(\bar{X}_1 - \bar{\bar{X}}) = 2$	$(\bar{X}_1 - \bar{\bar{X}}) = -1$

$\bar{\bar{X}} = 2$

So what is the variance between our treatment means?

$$MS_{between} = \frac{SS_{between}}{df_{between}}$$

We need to look at the deviation between each treatment mean and the "grand mean"

The Logic of ANOVA

Treatment 1 50°	Treatment 2 70°	Treatment 3 90°
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$
$(\bar{X}_1 - \bar{\bar{X}})^2 = 1$	$(\bar{X}_1 - \bar{\bar{X}})^2 = 4$	$(\bar{X}_1 - \bar{\bar{X}})^2 = 1$

$\bar{\bar{X}} = 2$

So what is the variance between our treatment means?

$$MS_{between} = \frac{SS_{between}}{df_{between}}$$

We need to square those deviations

The Logic of ANOVA

Treatment 1 50°	Treatment 2 70°	Treatment 3 90°
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$
$n_1(\bar{X}_1 - \bar{\bar{X}})^2 = 5$	$n_2(\bar{X}_2 - \bar{\bar{X}})^2 = 20$	$n_3(\bar{X}_3 - \bar{\bar{X}})^2 = 5$

$\bar{\bar{X}} = 2$

So what is the variance between our treatment means?

$$MS_{between} = \frac{SS_{between}}{df_{between}}$$

And weight them by the sample size of each treatment

The Logic of ANOVA

Treatment 1 50°	Treatment 2 70°	Treatment 3 90°
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$
$n_1(\bar{X}_1 - \bar{\bar{X}})^2 = 5$	$n_2(\bar{X}_2 - \bar{\bar{X}})^2 = 20$	$n_3(\bar{X}_3 - \bar{\bar{X}})^2 = 5$

$\bar{\bar{X}} = 2$

So what is the variance between our treatment means?

$$MS_{between} = \frac{SS_{between}}{df_{between}} = \frac{\sum_{j=1}^{k} n_j(\bar{X}_j - \bar{\bar{X}})^2}{df_{between}}$$

Now add them up to get $SS_{between}$

and put your $df_{between}$ in the denominator

The Logic of ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$
$n_1(\bar{X}_1 - \bar{\bar{X}})^2 = 5$	$n_2(\bar{X}_2 - \bar{\bar{X}})^2 = 20$	$n_3(\bar{X}_3 - \bar{\bar{X}})^2 = 5$

$\bar{\bar{X}} = 2$

So what is the variance between our treatment means?

$$MS_{between} = \frac{SS_{between}}{df_{between}} = \frac{\sum_{j=1}^{k} n_j(\bar{X}_j - \bar{\bar{X}})^2}{k-1}$$

The Logic of ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$

$\bar{\bar{X}} = 2$

So what is our test statistic?

$$MS_{between} = \frac{SS_{between}}{df_{between}} \qquad MS_{within} = \frac{SS_{within}}{df_{within}}$$

$$F(df_b, df_w) = \frac{MS_{between}}{MS_{within}}$$

The Logic of ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$

$\bar{\bar{X}} = 2$

What are its degrees of freedom?

$$df_b = df_{between} = k - 1$$

$$df_w = df_{within} = N - k$$

Tying the Logic of ANOVA back with the theory

F statistic

$$F = \frac{\text{obtained variance between sample means}}{\text{variance expected by chance}}$$

⬇

$$F = \frac{\text{between-treatment variance}}{\text{within-treatment variance}}$$

⬇

$$F(df_b, df_w) = \frac{MS_{between}}{MS_{within}} = \frac{\sum_{j=1}^{k} n_j(\bar{X}_j - \bar{\bar{X}})^2 / df_b}{\sum_{j=1}^{k} SS_j / df_w} \approx \frac{n_j \sigma^2_{\bar{X}_j}}{\sigma}$$

Formulas for ANOVA (for you to memorize)

Between-group statistics

$$SS_{between} = \sum_{j=1}^{k} n_j(\bar{X}_j - \bar{\bar{X}})^2 \qquad df_{between} = k - 1 \qquad MS_{between} = SS_{between} / df_{between}$$

Within-group statistics

$$SS_{within} = \sum_{j=1}^{k} SS_j = \sum_{j=1}^{k}\sum_{i=1}^{n_j}(X_i - \bar{X}_j)^2 \qquad df_{within} = N - k \qquad MS_{within} = SS_{within} / df_{within}$$

Total statistics

$$SS_{Total} = SS_{between} + SS_{within} = \sum_{i}^{N}(X_i - \bar{\bar{X}})^2 \qquad df_{Total} = df_{between} + df_{within} = N - 1$$

F-statistic

$$F(df_{between}, df_{within}) = MS_{between} / MS_{within}$$

Effect size estimate

$$r^2 = \eta^2 = \frac{SS_{between}}{SS_{Total}}$$

Steps in ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$

$\bar{\bar{X}} = 2$

Step 1: Calculate each treatment mean and the grand mean

Steps in ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º	
0	4	1	
1	3	2	
3	6	2	$\bar{\bar{X}} = 2$
1	3	0	
0	4	0	
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$	
$SS_1 = 6$	$SS_2 = 6$	$SS_3 = 4$	

Step 2: Calculate the MS_{within} $= \frac{6+6+4}{12} = 1.33$

a) Get the SS for each treatment

b)

$$MS_{within} = \frac{SS_w}{df_w} = \frac{\sum_{j=1}^{k} SS_j}{N-k}$$

Steps in ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º	
0	4	1	
1	3	2	
3	6	2	$\bar{\bar{X}} = 2$
1	3	0	
0	4	0	
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$	
$n_1(\bar{X}_1 - \bar{\bar{X}})^2 = 5$	$n_2(\bar{X}_2 - \bar{\bar{X}})^2 = 20$	$n_3(\bar{X}_3 - \bar{\bar{X}})^2 = 5$	

Step 3: Calculate the $MS_{between}$

a) For each treatment mean, find the squared deviation from the grand mean, and multiply by the treatment n.

b)

$$MS_{between} = \frac{SS_b}{df_b} = \frac{\sum_{j=1}^{k} n_j(\bar{X}_j - \bar{\bar{X}})^2}{k-1} \qquad = \frac{5+20+5}{2} = 15$$

Steps in ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º	
0	4	1	
1	3	2	
3	6	2	$\bar{\bar{X}} = 2$
1	3	0	
0	4	0	
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$	

Step 4: Calculate F

$$F(2,12) = \frac{MS_{between}}{MS_{within}} = \frac{15}{1.33} = 11.28$$

Steps in ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º	
0	4	1	
1	3	2	
3	6	2	$\bar{\bar{X}} = 2$
1	3	0	
0	4	0	
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$	

Step 5: Compare F value to F_{crit}. If $F > F_{crit}$, reject the null hypothesis

$F(2,12) = 11.28 \quad F_{crit} = 3.88$

$F > F_{crit}$ Reject the null!

Steps in ANOVA

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º	
0	4	1	
1	3	2	
3	6	2	$\bar{\bar{X}} = 2$
1	3	0	
0	4	0	
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$	

Step 6: Obtain an effect size estimate – how “big” is the effect

$$r^2 = \eta^2 = \frac{SS_{between}}{SS_{Total}} = \frac{SS_{between}}{SS_{between} + SS_{within}} = \frac{30}{30+16} = .65$$

“65% of the variation in learning performance in this study is explained by temperature”

ANOVA summary table

Source	SS	df	MS	
Between treatments	$\sum_{j=1}^{k} n_j(\bar{X}_j - \bar{\bar{X}})^2$	k - 1	SS_b / df_b	$MS_{between} / MS_{within}$
Within treatments	$\sum_{j=1}^{k} SS_j$	N - k	SS_w / df_w	
Total	$\sum_{i}^{N} (X_i - \bar{\bar{X}})^2$	N - 1		

ANOVA summary table

Treatment 1 50º	Treatment 2 70º	Treatment 3 90º
0	4	1
1	3	2
3	6	2
1	3	0
0	4	0
$\bar{X}_1 = 1$	$\bar{X}_2 = 4$	$\bar{X}_3 = 1$

$SS_{total} = 46$

$SS_{between} = 30$ $\quad$ $SS_{within} = 16$

Source	SS	df	MS	
Between treatments	30	2	15	F = 11.28
Within treatments	16	12	1.33	
Total	46	14		

The F Distribution

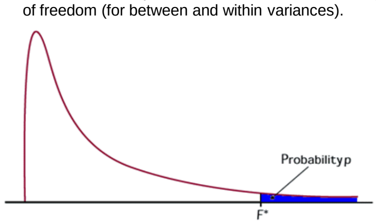
F statistics

- Because F-ratios are computed from two variances, F values will always be positive numbers.
- When H_0 is true, the numerator and denominator of the F-ratio are measuring the same variance. In this case the two sample variances should be about the same size, so the ratio should be near 1. In other words, the distribution of F-ratios should pile up around 1.00.
- H_0 rejected only from large F values, but the F-test tests two-tailed hypotheses about mean differences (they can be in any direction).

The F Distribution

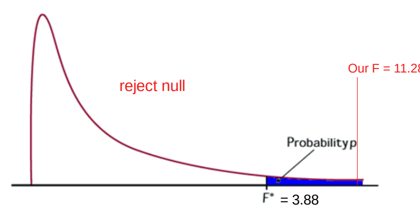
F Distribution

- Positively skewed distribution: Concentration of values near 1, no values smaller than 0 are possible
- Like t, F is a family of curves. Need to use both degrees of freedom (for between and within variances).



The F Distribution

So for an alpha of .05, the critical value of F with degrees of freedom 2,12 is



Assumptions

1. **The samples are independent simple random samples**
2. **The populations are normal**
3. **The populations have equal variance**

Exercise – by hand

The data depicted below were obtained from an experiment designed to measure the effectiveness of three pain relievers (A, B, and C). A fourth group that received a placebo was also tested.

Placebo	Drug A	Drug B	Drug C
0	0	3	8
0	1	4	5
3	2	5	5

Is there any evidence for a significant difference between groups?

Exercise – by hand

Placebo	Drug A	Drug B	Drug C
0	0	3	8
0	1	4	5
3	2	5	5

Source	SS	df	MS	
Between treatments				
Within treatments				
Total				

Exercise – by hand

Placebo	Drug A	Drug B	Drug C
0	0	3	8
0	1	4	5
3	2	5	5

Source	SS	df	MS	
Between treatments	54	3	18	F = 9
Within treatments	16	8	2	
Total	70	11		

F_{crit} = 4.07, so reject null

Comparing F and t

What is the relationship between F and t?

Placebo	Drug C
0	8
0	5
3	5

1. Compute an independent-samples t statistic for this data t = -3.54
2. Compute an F statistic for this data F = 12.53

Comparing F and t

What is the relationship between F and t?

$F = t^2$